1.4 Relations and Functions
A relation is a correspondence between two sets. If x and y are two elements in these sets and if a relation exists between x and y, then x corresponds to y, or y depends on x.

DEFINITION OF A FUNCTION:
Let X and Y two nonempty sets. A function from X into Y is a relation that associates with each element of X, exactly one element of Y. However, an element of Y may have more than one elements of X associated with it.
That is for each ordered pair (x,y), there is exactly one y value for each x, but there may be multiple x values for each y. The variable x is called the independent variable (also sometimes called the argument of the function), and the variable y is called dependent variable (also sometimes called the image of the function.)

Below is the graph of y=x^2-4  (an upward parabola with vertex (0,-4))

For y=12, there are two possible x’s. x=-4, and x=4.

VERTICAL-LINE TEST THEOREM
A set of points in the xy-plane is the graph of a function if and only if (iff), every vertical line intersects the graph in at most one point.
\( x = y^2 \) is not a function from \( X \) into \( Y \), because there is not exactly one \( y \) value for each \( x \). Solving for \( y \), you get \( y = \pm \sqrt{x} \)

which means there are two possible values for \( y \).

This figure is a parabola with vertex at origin, and which axis of symmetry is with the x-axis, and opens to the right.

**Does this graph pass the vertical lines test?**

**Can you think of any other equations that are NOT functions of \( x \)?**

**A circle?**
DOMAIN AND RANGE
The set $X$ is called the **domain** of the function. This is the set of all possible $x$ values specified for a given function.

The set of all $y$ values corresponding to $X$ is called the **range**.

In the example below, we see that $x$ goes off into infinity in both directions, so the domain of $y=x^2$ is

{all real numbers}

However, we see there are no corresponding values of $y$ that are less than -4, so the range is \{y | y≥-4\}

Example 4 p. 36
Consider the equation

$y = 2x - 5$, where the domain is \{x|1 ≤ x ≤ 6\}

Is this equation a function?
Notice that for any $x$, you can only get one answer for $y$.
(E.g. for $x = 1$, $y = 2(1) - 5 = -3$.) Therefore the equation is a function.

What is the range?
Since this is a straight line, we need only check $y$ values at endpoints of domain. The $y$ values are also called function values, so they are often referred to as $f(x)$, which means the value of the function at $x$ (not $f$ times $x$).

The endpoints of the domain are 1 and 6.

$f(1) = 2(1) - 5 = -3$
$f(6) = 2(6) - 5 = 7$

So the range is \{y|-3 ≤ y ≤ 7\}
A function, \( f \), is like a machine that receives as input a number, \( x \), from the domain, manipulates it, and outputs the value, \( y \). The function is simply the process that \( x \) goes through to become \( y \). This “machine” has 2 restrictions:
1. It only accepts numbers from the domain of the function.
2. For each input, there is exactly one output (which may be repeated for different inputs).

Finding Values of a Function
Example 5 p. 38
For the function \( f \) defined by \( f(x) = 2x^2 - 3x \), evaluate

b) \( f(x) + f(3) = \left[ 2x^2 - 3x \right] + \left[ 2(3)^2 - 3(3) \right] \\
   = 2x^2 - 3x + 18 - 9 \\
   = 2x^2 - 3x + 9 \)

e) \( f(x+3) = 2(x+3)^2 - 3(x+3) \\
   = 2(x^2 + 6x + 9) - 3x - 9 \\
   = 2x^2 + 12x + 18 - 3x - 9 \\
   = 2x^2 + 9x + 9 \)

Notice that \( f(x) + f(3) \) does not equal \( f(x+3) \)
Difference Quotient of $f$

$$\frac{f(x+h) - f(x)}{h} =$$

$$= \frac{[2(x+h)^2 - 3(x+h)] - [2x^2 - 3x]}{h}$$

$$= \frac{[2(x^2 + 2hx + h^2) - 3x - 3h] - [2x^2 - 3x]}{h}$$

$$= \frac{2x^2 + 4hx + 2h^2 - 3x - 3h - 2x^2 + 3x}{h}$$

$$= \frac{4hx + 2h^2 - 3h}{h}$$

$$= \frac{h(4x + 2h - 3)}{h}$$

$$= 4x + 2h - 3$$

This is called the *difference quotient of $f$*, which is an important function in calculus. In calculus, the derivative, $dy/dx$, is defined as the limit of this function as $h$ approaches 0.

**IMPORTANT FACTS ABOUT FUNCTIONS**

1. For each $x$ in the domain of a function $f$, there is one and only one image $f(x)$ in the range.

2. $f$ is the symbol that we use to denote the function. It is symbolic of the equation that we use to get from an $x$ in the domain to the $f(x)$ in the range.

3. If $y = f(x)$, then $x$ is called the **independent variable** or argument of $f$, and $y$ is called the **dependent variable** or the value of $f$ at $x$ (or the image of $f$ at $x$).
Example 8 p. 40
Find the domain of each of the following functions:

b) \( g(x) = \frac{3x}{x^2 - 4} \)  
c) \( h(t) = \sqrt{4 - 3t} \)

The domain is the set of all possible \( x \) values that can be used in these functions.

b) \( g(x) \) is the division of \( 3x \) by \( x^2 - 4 \). This is undefined if the denominator is 0, so we have the limitation that \( x^2 - 4 \neq 0 \).

Solve for \( x \) to find specifications for what \( x \) cannot be.
\[ x^2 \neq 4 \]
\[ x \neq \pm 2 \]

Therefore domain is \( \{x | x \neq \pm 2\} \) The function \( g(x) \) is not defined at \( x=2 \) or \( x=-2 \).

c) \( h(t) \) is the square root of \( 4 - 3t \). Only nonnegative numbers have real square roots, so the expression on the radical must be \( \geq 0 \).
\[ 4 - 3t \geq 0 \]
\[ -3t \geq -4 \]
Remember when you multiply an inequality by a negative number, the inequality reverses.
\[ -3t/(-3) \leq -4/(-3) \]
\[ t \leq -4/3 \]

Therefore domain is \( \{t | t \leq -4/3\} \)

Another way to state this is in interval form: \( \left( -\infty, -\frac{4}{3} \right] \)

This is not a coordinate point. It’s just another way to describe a set of numbers.

NOW YOU DO #37 on p.46
Look at the graph to the right (\( y=1/x \)):

Is this graph a function?
Yes, because a vertical line through any \( x \)-value on the graph only intersects the graph once.

What are the domain and range?
The domain (possible \( x \) values) is \( \{x | x \neq 0\} \)
The range (possible \( y \) values) is \( \{y | y \neq 0\} \)
Problem 48 on p. 47

a) Find $f(0)$ and $f(6)$
What is $y$ when $x$ is 0 and $x$ is 6? From the data given, we see the y-coordinate at $x=0$ is 0, so $f(0)=0$. The y-coordinate at $x=6$ is also 0, so $f(6)=0$.

b) Find $f(2)$ and $f(-2)$
What is $y$ when $x$ is 2 and $x$ is -2? From the data given, we see the y-coordinate at $x=2$ is -2, so $f(2)=-2$. The y-coordinate at $x=-2$ is 1, so $f(-2)=1$.

c) Is $f(3)$ positive or negative? We see that at $x=3$ the graph is below the x-axis (where $y < 0$) so $f(3)$ is negative.

d) Is $f(-1)$ positive or negative? We see that at $x=3$ the graph is below the x-axis (where $y < 0$) so $f(3)$ is negative.

e) For what numbers $x$ is $f(x) = 0$? In other words, at which values of $x$ cross the x-axis (where $y=0$)? The graph crosses the x-axis at $x=0, x=4, x=6$.

f) For what numbers $x$ is $f(x) < 0$? In other words, at which values of $x$ is the graph below the x-axis? Remember, the coordinates where $y=0$ are not included. The graph is $< 0$ only for $0 < x < 4$. In interval form this is $(0,4)$.

g) What is the domain of $f$? Domain is the possible x values. Remember that this graph does not continue into infinity on both sides. It is only define for the graph drawn. Therefore, can infer that the possible x values are $-4 \leq x \leq 6$, or $[-4,6]$

h) What is the range of $f$? The y values range from as low as -2 to as high as 3, so range is $\{y\mid -2 \leq y \leq 3\}$.

i) What are the x-intercepts? The x-intercepts are found when $y=0$, which are the points $\{(0,0), (4,0),(6,0)\}$.

j) What is the y-intercept? By definition, this would not be a function if it crossed the y-axis (or any other vertical line) more than once. The only point that does this is $(0,0)$.

k) How often does the line $y=-1$ intersect the graph? If we draw a horizontal line through $y=-1$, we’d see it intersects twice.

l) How often does the line $x=1$ intersect the graph? Three times.

m) For what value of $x$ does $f(x) = 3$? Remember $f(x)$ is the same as $y$. What is $x$ when $y=5$? There’s only one point on the graph that gives a y-value of 3. That is when $x=5$.

n) For what value of $x$ does $f(x)=-2$? There’s only one point on the graph that gives a y-value of -2. That is when $x=2$. 
Example 11 on p. 44

\[ f(x) = \frac{x}{x + 2} \]

a) Is the point \((1, \frac{1}{2})\) on the graph of \(f\)? Substitute 1 for \(x\) and \(\frac{1}{2}\) for \(f(x)\) and see if the statement is true.
Does \(\frac{1}{2} = \frac{1}{1+2}\) ? \(\frac{1}{2} \neq \frac{1}{3}\) Therefore \((1, \frac{1}{2})\) is not on the graph.

b) If \(x = -1\), what is \(f(x)\)? \(f(-1) = -1/(-1+2) = -1/1 = -1\)
The point at \(x = -1\) is \((-1,-1)\).

c) If \(f(x) = 2\), what is \(x\)? YOU DO THIS?

Example 12 on p.45 Area of a Circle

\[ A(r) = \pi r^2 \]

where \(r\) represents the radius of the circle. The domain is \(\{r | r > 0\}\). Why?

NOW YOU DO #87 on p.50

HOMEWORK

p. 46  #9, 17, 25, 29, 39, 45, 47, 65, 73, 85