5.3 THE COMPLEX PLANE

Complex Numbers are numbers of the form $z = x + yi$, where $a$ and $b$ are real numbers. The real number $x$ is called the real part of the number $z$; the real number $y$ is called the imaginary part of $z$.

What is $i$?

$i^2 = -1$

so

$i = \sqrt{-1}$ which is why $i$ is imaginary!

The magnitude (or modulus) of $z$ is denoted by $|z|$ and defined as the distance from the origin to the point $(x, y)$. That is,

$$|z| = \sqrt{x^2 + y^2}$$

The conjugate of $z$ is $\overline{z} = x - yi$

so

$$\overline{z}z = (x + yi)(x - yi)$$

$$= x^2 - xyi + xyi - y^2i^2$$

$$= x^2 - y^2(-1) = x^2 + y^2$$

so

$$\sqrt{\overline{z}z} = \sqrt{x^2 + y^2} = |z|$$
POLAR FORM OF A COMPLEX NUMBER

Recall that \(x = r \cos \theta\) and \(y = r \sin \theta\)

If we let \(r \geq 0\) and \(0 \leq \theta \leq 2\pi\), then the complex number \(z = x+yi\) can be written in polar form as

\[z = x + yi = (r \cos \theta) + (r \sin \theta)i = r(\cos \theta + i \sin \theta)\]

In polar form, \(\theta\) is called the argument of \(z\). Since \(r^2 = x^2 + y^2\)

\[|z| = r = \sqrt{x^2 + y^2}\]

is the modulus (or magnitude) of \(z\).

\(\theta\) is the angle formed with the positive x-axis, which can be found in the following way:

\(\theta = \tan^{-1}(y/x)\) if (x,y) is in Quadrant I or IV

\(\theta = \tan^{-1}(y/x) + \pi\) if (x,y) is in Quadrant II or III

(because \(\tan^{-1}\) only gives answers in Q I or IV we need to add \(\pi\) if the coordinates are actually in a different Quadrant.

Example 1 on p.347

Plot \(z = \sqrt{3} - i\) and put \(z\) in polar form.

What is \(x\)?
What is \(y\)?
What Quadrant is (x,y) in?
What is \(z\) in polar form?
First, find \(r\).
Then find \(\theta\), \(\theta = tan^{-1}(y/x)\)

Now you do #3 on p.353
The Product and Quotient of two Complex Numbers

\[ z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)] \]

If \( z_2 \neq 0 \), then

\[ \frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)] \]

Proof:  
\[ z_1 z_2 = [r_1 (\cos \theta_1 + i \sin \theta_1)] [r_2 (\cos \theta_2 + i \sin \theta_2)] \]
\[ = r_1 r_2 [(\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2)] \]
\[ = r_1 r_2 [\cos \theta_1 \cos \theta_2 + i \sin \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 + \cos \theta_2 \sin \theta_1] \]
\[ = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \]

Using our sum of angle formulas we get:
\[ = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)] \]

In a similar way, division of two complex numbers can be proven*.

Example 3
If \( z = 3(\cos 20^\circ + i \sin 20^\circ) \) and \( w = 5(\cos 100^\circ + i \sin 100^\circ) \)

Find the following (leave your answers in polar form):

a) \( zw \)

\[ zw = (3*5)[\cos(20^\circ + 100^\circ) + i \sin(20^\circ + 100^\circ)] \]
\[ = 15[\cos(120^\circ) + i \sin(120^\circ)] \]

b) \( \frac{z}{w} \)

\[ \frac{z}{w} = \frac{3}{5} [\cos(20^\circ - 100^\circ) + i \sin(20^\circ - 100^\circ)] \]
\[ = \frac{3}{5} [\cos(-80^\circ) + i \sin(-80^\circ)] \]
\[ = \frac{3}{5} [\cos(80^\circ) + i \sin(280^\circ)] \]

* To see more on this see: http://scholar.hw.ac.uk/site/maths/topic17.asp?outline=no
Multiplying two complex numbers in polar form is easily undertaken by using the rules set out in the last section. This leads to a very simple formula for calculating powers of complex numbers - known as De Moivre's theorem.

Consider the product of \( z \) with itself \( (z^2) \) if \( z = r(\cos \theta + i \sin \theta) \)

\[
z^2 = z \cdot z = [r(\cos \theta + i \sin \theta)] [r(\cos \theta + i \sin \theta)]
\]

\[
= r^2 [\cos (\theta + \theta) + i \sin (\theta + \theta)]
\]

\[
= r^2 [\cos (2\theta) + i \sin (2\theta)]
\]

Now consider \( z^3 \)

Take \( z^2 = r^2(\cos 2\theta + i \sin 2\theta) \) and multiply this by \( z = r(\cos \theta + i \sin \theta) \)

This gives \( z^3 = [r^2(\cos 2\theta + i \sin 2\theta)][r(\cos \theta + i \sin \theta)] = r^3(\cos 3\theta + i \sin 3\theta) \)

A pattern has emerged.

This result can be extended to the \( n \)th power and is known as **De Moivre's Theorem**.

\[
\text{De Moivre's theorem}
\]

**If** \( z = r(\cos \theta + i \sin \theta) \) **, then**

\[
z^n = r^n[\cos (n\theta) + isin(n\theta)] \quad \text{where} \quad n \geq 1 \text{ is a positive integer.}
\]

This is a very useful result as it makes it simple to find \( z^n \) once \( z \) is expressed in polar form.

**Example 4**

Write \( [2(\cos 20^\circ + isin 20^\circ)]^3 \) in the standard form \( a+bi \)

We want to simplify \( z^3 \), where \( z = 2(\cos 20^\circ + isin 20^\circ) \)

In this polar form of \( z \), \( r = 2 \) and \( \theta = 20^\circ \).

Using DeMoivre’s Theorem, we can rewrite \( z^3 \):

\[
= 2^3[\cos(3*20^\circ) + isin(3*20^\circ)]
\]

\[
= 8[\cos(60^\circ) + isin(60^\circ)]
\]

\[
= 2^3[\cos(3\cdot20^\circ) + i\sin(3\cdot20^\circ)]
\]

\[
= 8[\cos(60^\circ) + i\sin(60^\circ)]
\]

\[
= 8\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right] = 4 + (4\sqrt{3})i
\]

Now you do #31 on p. 353
Finding Complex Roots

Let \( w = r (\cos \theta_0 + i \sin \theta_0) \) be a complex number and let \( n \geq 2 \) be an integer. If \( w \neq 0 \), there are \( n \) distinct complex roots of \( w \), were the \( k \)th root is:

\[
 z_k = \sqrt[n]{r} \left[ \cos \left( \frac{\theta_0 + 2k\pi}{n} \right) + i \sin \left( \frac{\theta_0 + 2k\pi}{n} \right) \right]
\]

and \( k = 0, 1, 2, 3, \ldots, n-1 \).

Example 6

Finding Complex Cube Roots

Find the complex cube roots of \( z = -1 + \sqrt{3}i \)

What that are asking for is \( \sqrt[3]{z} \), which will give you three possible answers: \( z_0, z_1, z_2 \)

First, convert to polar form.

\[
r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1 + 3} = 2
\]

The complex plane coordinates are \((-1, \sqrt{3})\) which is in Q2, so \( \cos \theta < 0 \) and \( \sin \theta > 0 \)

\[
\theta = \tan^{-1} \left( \frac{y}{x} \right) = \tan^{-1} \left( \frac{\sqrt{3}}{-1} \right) = 120^\circ
\]

In polar form, \( z = 2(\cos 120^\circ + i \sin 120^\circ) \)

The rest will be done in class....
HOMEWORK

P. 353  #3, 7, 11, 13, 17, 23, 27, 29, 31, 41, 43